Lie Algebra Representations Cheat Sheet

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Concept	For Mathemeticians	For Physicists	Concretely
Lie Group	A group that is also a differentiable manifold	A closed set of differentiable transformations	$U(N), O(N), Sp(N)$
Lie Algebra	A vector space $\mathfrak{g}$ with an antisymmetric bilinear bracket relation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$	The tangent space (i.e. derivative) of a Lie group; the elements (of which there are finitely many) are the generators of the group so algebras are a great way to understand the group	$\mathfrak{u}(N), \mathfrak{o}(N), \mathfrak{sp}(N)$
Representation	A homomorphism ρ from the group G (or algebra $\mathfrak{g}$) to the space of general linear operators (matrices)	A set of matrices that under multiplication (or commutation) exhibit the same relations as the abstract group (or algebra); a way of making group relations concrete.	The set of $\{e^{i\phi} \vert \forall \phi \in [0, 2\pi)\}$ is a representation of $U(1)$
Irreducible Representation (or irrep)	A nonzero representation with no proper subrepresentation	A representation that isn't made of any others; the building blocks of representations	Counterexample: The set of matrices $\{\text{diag}(e^{i\phi}, e^{2i\phi})\}$ is *not* an irrep.
Adjoint Representation of a Lie Group (Ad)	A representation of a Lie group as linear transformations of the Lie algebra conjugated by an element of the group; $\text{Ad}_g \in \text{Aut}(\mathfrak{g})$.	A particularly important representation of a Lie group in terms of how it transforms its own Lie algebra	$\text{Ad}_g(X) = gXg^{-1}$
Adjoint Representation of a Lie Algebra (ad)	A representation of a Lie algebra as linear transformations of the Lie algebra	A particularly important representation of a Lie algebra as essentially the action of the algebra on itself. It both provides linear operations upon elements of the algebra that give the values of the bracket relation and is a representation of the algebra. Its weights are the roots.	$\text{ad}_X(Y) = [X, Y]$; $[\text{ad}_X, \text{ad}_Y] = \text{ad}_{[X,Y]}$; $[\text{ad}_{L_i}]^k{}_j = i f_{ij}{}^k$
Structure Constants	Completely antisymmetric indexed objects that define the bracket relation of a Lie algebra	The coefficients that describe the bracket relation, which defines the action of the algebra, but also the data of the adjoint representation.	$[L_i, L_j] = -i f_{ij}{}^k L_k$

Quadratic Casimir Operator	For a given representation of an algebra and a given basis equipped with a product, $\widehat{\Omega} = -f^{i}_{kl} f^{klj} \widehat{L}_i \widehat{L}_j$	A particular object in a representation of an algebra that commutes with all other elements, and so can be made proportional to the identity matrix; its eigenvalue depends on the representation.	In the hydrogen atom (i.e. $SU(3)$ representations), L^2, J^2, S^2.
Cartan Subalgebra	The maximal subalgebra that can be simultaneously diagonalized in the representation	The biggest subalgebra that we can extract to find simultaneous eigenvectors; their eigenvalues are the weights	$SU(2)$: L_z, J_z, S_z $SU(3) : \lambda_3, \lambda_8$
Weights and Weight Vectors	For an algebra over a field, a weight a homomorphism from the algebra to the field; the weight vector is the vector of those weights.	The weight vector is the vector of the Cartan subalgebra's eigenvalues of a given eigenvector in a representation. It says "where" an object in a representation lives in weight space.	$SU(2)$: m_l, m_j, m_s $SU(3)$: The quark triplet and the meson octet + singlet
Roots and Root Systems	A root is a vector of coefficients from the action of the bracket relation on the Cartan subalgebra and some basis of the remaining quotient algebra	The action of the ladder operators on the weights. The roots and root system form a "compass rose" along which we can travel around the lattice of weight space. These are a generalization of how we think about ladder operators.	$SU(2)$: ± 1 $SU(3)$: $\pm(1,0)$, $\pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
Gell-Man Matrices	A particular representation (the defining representation) of $SU(3)$, analogous to the Pauli matrices for $SU(2)$	$\widehat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \widehat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \widehat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$ $\widehat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \widehat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$ $\widehat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \widehat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \widehat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	
Fundamental Representation	A representation whose highest weight is the fundamental weight (tautological as it may seem, there is math™ behind this)	As far as intuition is usually concerned, this tends to be the smallest faithful (i.e. nontrivial) representation of an algebra.	$SU(2)$: $j = \frac{1}{2}$ $SU(3)$: $\mathbf{3}$
Classification of Semisimple Lie Algebras	Semisimple Lie algebras can be classified by the root system associated with their Cartan subalgebras into four families, with five exceptional algebras.	All root systems belong in one of four families based on the angles between their roots (except for five exceptions).	A_n : $\mathfrak{sl}(n+1)$; B_n : $\mathfrak{so}(2n+1)$; C_n : $\mathfrak{sp}(2n)$; D_n : $\mathfrak{so}(2n)$