

# Dirac brackets and constrained quantization

## 1 Introduction and motivation

In the simple quantum mechanical or quantum field theoretic settings, we learned the canonical ways of quantization through Poisson brackets. However, just as in classical dynamics, we might often times have many theories that are easier to describe with more dynamical variables than the actual degrees of freedom, and we have to deal with constraints. In typical models of quantum theories, this happens because of the easiness in implementing symmetry principles in the Lagrangian formalism and the corresponding Hamiltonian formalism (which is required for quantization) might be constrained.

The important class of problems are gauge theories, in which there are generally large amount of redundancies that we have to handle by systematically imposing constraints. Gravity is in also a class of more general “gauge theories” in the sense that they deal with invariance under local transformations, namely general covariance.

Some examples are in order. The most classic one is the relativistic point particle. The familiar worldline action is the one proportional to the worldline proper length

$$S_{pp} = -m \int d\tau = \int d\sigma \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}}, \quad (1)$$

where  $d\tau = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$  with some reasonable parametrization of the worldline  $\sigma$ . There is an equivalent action in the “einbein formulation”

$$S_{pp} = -\frac{1}{2} \int d\sigma [e^{-1} \dot{x}^2 + em^2], \quad (2)$$

where  $e(s)$  is the einbein, an auxiliary degree of freedom. By substituting the equation of motion for this variable, we can get back to Eq. (1). There is yet another equivalent action in phase space

$$S_{pp} = - \int d\sigma \left[ \dot{x} \cdot p - \frac{1}{2} e(p^2 - m^2) \right]. \quad (3)$$

This is equivalent to Eq. (2) by eliminating the canonical momentum. We observe firstly that in this form of the action, the auxiliary field is essentially a Lagrange multiplier which imposes the mass shell constraint (classically)  $p^2 - m^2 = 0$  to this particle. Secondly, the einbein term acts as the “Hamiltonian” (c.f. Legendre transform  $L = \sum_i \dot{q}_i p_i - H$ ).

A less familiar example is the (closed bosonic) string theory. The stringy counterpart to Eq. (1) is the Nambu-Goto action which is proportional to the worldsheet proper area

$$S_{NG} = -T \int d\sigma^1 \oint d\sigma^2 \sqrt{\left| \det \left[ g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \right] \right|} = -T \int d^2\sigma \sqrt{(\partial_1 X \cdot \partial_2 X)^2 - (\partial_1 X)^2 (\partial_2 X)^2}.$$

For this problem, we introduce an auxiliary worldsheet metric  $h_{ab}$  (the einbein is, in a sense, the metric on the worldline). Now, the action can be reformulated as the so called Polyakov action (who used path integral methods to do the

quantization)

$$S_{Polyakov} = -\frac{1}{2}T \int d^2\sigma \sqrt{|\det h|} h^{ab} \partial_a X \cdot \partial_b X.$$

There are two old ways of quantizing this, one is by choosing the so called “light-cone gauge” strictly fixing  $h_{ab}$  (this is basically the straight forward way of removing redundancies). The alternative approach is the “old covariant quantization” (it is called old since there is a more modern covariant technique called BRST quantization). In this picture, the conditions on  $h_{ab}$  is much weaker, and we still have to impose additional constraints by requiring that the variations with respect to the auxiliary metric vanishes. Once again, it can be shown that the alternative way of writing the action in the phase space formulation is

$$S_{phase} = - \int d^2\sigma \{ \dot{X} \cdot P - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+ \},$$

where the  $\lambda$ s are Lagrange multipliers and  $\mathcal{H}$ s are precisely the so called “Virasoro constraints”. The Virasoro algebra associated with the constraints is a signature of a 2-dimensional conformal field theory as is well known.

Apart from the final example, we can see that in generic constrained systems, constraints naturally appear in the form of Hamiltonians. Here, we would like to explore Dirac’s conjecture on quantization of such theories. The idea originated from Dirac’s seminal paper in 1950 [1], and was delivered in a series of lectures in 1964 [2]. We will mainly follow the lectures and the exposition given by Weinberg [3]. A more comprehensive reference specializing on constrained quantization can be found in [4, 5].

## 2 Dirac’s formulation of constrained quantization

### 2.1 Revision of Poisson brackets and quantization

In simple Hamiltonian dynamical systems, we have the canonical variables  $Q_a$  and  $P_a$ . The Poisson bracket is defined as (summations assumed)

$$[A, B]_P = \frac{\partial A}{\partial Q^a} \frac{\partial B}{\partial P_a} - \frac{\partial A}{\partial P_a} \frac{\partial B}{\partial Q^a}.$$

The algebra is identical to commutators, namely satisfying antisymmetry, Leibniz rule, and Jacobi identity

$$[A, B]_P = -[B, A]_P,$$

$$[A, BC]_P = [A, B]_P C + B[A, C]_P,$$

$$[A, [B, C]_P]_P + [B, [C, A]_P]_P + [C, [A, B]_P]_P = 0.$$

The canonical Poisson bracket is given by  $[Q^a, P_b]_P = \delta_b^a$ . Comparing to canonical commutator  $[Q^a, P_b] = i\delta_b^a$ , we quantize by

$$[A, B] = i[A, B]_P.$$

### 2.2 Constraints and their classification

With constraints, the subtlety is that naïve commutators might not necessarily be consistent; thus we have to carefully account for them while quantizing the theory. We first observe that constraints can behave differently, so we need to classify them first.

Following Dirac’s conventions, we call the explicit relations between these canonical variables “primary constraints”

$$\phi_M(Q, P) \approx 0,$$

where in the following this curly equality means valid after imposing constraints. Since they are zero, we are free to add these to the Hamiltonian

$$H + u_M \phi_M,$$

with coefficients  $u_M$  undetermined. The time evolution defined by this is given by

$$\dot{f} = [f, H]_P + u_M [f, \phi_M]_P.$$

We may apply this to the constraint functions themselves  $f = \phi_M$ . Consistency requires that the time evolution also vanishes after imposing the constraints. Such conditions may (or may not) yield additional conditions. The ones independent of the coefficients are defined as “secondary”

$$\varphi_M(Q, P) \approx 0.$$

And the same analysis can be applied to these leading to “secondary secondary” constraints. By definition, any ones that are left impose consistency conditions on these coefficients themselves. Therefore, after imposing constraints, we have a polynomial equation for  $u_M$  which we might solve in terms of  $Q$ s and  $P$ s

$$U_M(Q, P).$$

We may add to this solution any solution to

$$V_{aM}[\phi_{M'}, \phi_M]_P = 0,$$

where the first index denotes the possible independent solutions to this equation. Thus, the general solution to  $u_M$  satisfying all the conditions is given by

$$u_M = U_M + v_a V_{aM}.$$

In this form after all the consistency conditions has been satisfied, the  $v_a$  coefficients are arbitrary; they are the real “gauge freedom” of the theory. Going back to the point particle example, the einbein precisely corresponds to “reparametrization invariance”, the freedom to choose different parametrization of the worldline, and in the string example, the Virasoro constraints captures the worldsheet reparametrization symmetry.

More importantly, we also define any function of canonical variables (including the constraints) with vanishing Poisson brackets with the constraints to be first class. Otherwise, they are classified as second class. Now we may rewrite total Hamiltonian as

$$H + U_M \phi_M + v_a V_{aM} \phi_M = H' + v_a \phi'_a,$$

where we have defined

$$\begin{aligned} H' &= H + U_M \phi_M, \\ \phi'_a &= V_{aM} \phi_M. \end{aligned}$$

It can be shown by the definitions that both of these have vanishing Poisson brackets with all the constraints and are first class.

### 2.3 Dirac brackets and quantization

The analysis above tells us that the first class constraints are related to the real “gauge redundancies” as they do not affect the physical predictions of the system, and should be removed by gauge fixing. For the remaining second class constraints  $\chi_N = 0$ , by definition, the matrix  $C_{NM} = [\chi_N, \chi_M]_P$  is non-singular. The proposal by Dirac was a modified version of Poisson bracket which now we call the Dirac bracket

$$[A, B]_D = [A, B]_P - [A, \chi_N]_P (C^{-1})^{NM} [\chi_M, B]_P,$$

and the new quantization rule is simply

$$[A, B] = i[A, B]_D.$$

We first note that this new bracket algebra is constructed out of the old Poissonian one, and can be easily shown to have the same algebra (antisymmetry, Leibniz rule, and Jacobi identity). But now in addition, we also have

$$[\chi_N, B]_D = 0,$$

so that when we quantize this into commutators, the constraints are consistent.

To argue that this proposal is sensible, we look at a theorem proven by Maskawa and Nakajima. What they showed is that for any set of canonical pairs of variables, through a canonical transformation, it is always possible to separate into two pairs  $(Q^n, P_n)$  and  $(\mathcal{Q}^r, \mathcal{P}_r)$ , where the second pair are constraints  $\mathcal{Q}^r = \mathcal{P}_r = 0$ . Both are canonical variables so we still have to include all of them in computing Poisson brackets. We have Poisson brackets with any functions

$$[A, \mathcal{Q}^r]_P = -\frac{\partial A}{\partial \mathcal{P}_r}, \quad [B, \mathcal{P}_r]_P = \frac{\partial B}{\partial \mathcal{Q}^r},$$

and among the constraints

$$[\mathcal{Q}^r, \mathcal{P}_s]_P = \delta_s^r, \quad [\mathcal{Q}^r, \mathcal{Q}^s]_P = 0, \quad [\mathcal{P}_r, \mathcal{P}_s]_P = 0.$$

This tells us that the  $C$  matrix is block off-diagonal

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so the inverse is just  $-C$ . We can see that

$$\begin{aligned} [A, B]_D &= [A, B]_P + [A, \mathcal{Q}^r]_P [\mathcal{P}_r, B]_P - [A, \mathcal{P}_r]_P [\mathcal{Q}^r, B]_P \\ &= \frac{\partial A}{\partial Q^n} \frac{\partial B}{\partial P_n} - \frac{\partial B}{\partial Q^n} \frac{\partial A}{\partial P_n}, \end{aligned}$$

i.e. the Dirac brackets precisely captures the Poisson brackets with respect to the unconstrained variables.

### 3 Quantum electrodynamics in Coulomb gauge

For field theories, we simply extend the finite set of variables to the infinite set with field at each spatial position considered a degree of freedom, and promote partial derivatives into functional ones and sums to integrals. We consider the familiar case of the theory of quantum electrodynamics.

The quantum electrodynamics with charged sources is given by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu.$$

It is easy to see that time derivative of the field is absent, and the relevant dynamical degrees of freedoms in Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ , are the spatial parts  $A^i$  and their conjugates  $\Pi_j = \partial \mathcal{L} / \partial (\partial_0 A^j)$ . You might have encountered a guess of consistent canonical commutation relations for these reduced set of variables

$$[A^i(\mathbf{x}, t), \Pi_j(\mathbf{y}, t)] = i \int \frac{d^3 \mathbf{q}}{(2\pi)^3} e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{y})} \left[ \delta_j^i + \frac{q_i q_j}{\mathbf{q}^2} \right],$$

while all the others vanish.

Now we examine the problem more carefully in the Dirac formalism. The absence of time derivative of the field is implying a primary constraint

$$\Pi^0 = 0,$$

The secondary constraint associated with this is given by

$$\partial_i \Pi^i + J^0 = 0.$$

These can be shown to have vanishing Poisson brackets (when we still have the four pairs of fields) meaning they are first class. We will have to fix a gauge to eliminate the corresponding degrees of freedom. In Coulomb gauge, this secondary constraint gives

$$-\nabla^2 A^0 = J^0,$$

and the solution to this equation is given by

$$A^0(\mathbf{x}, t) = \int d^3 y \frac{J^0}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

Now we are left with the spatial degrees of freedom, but there are still second class constraints

$$\begin{aligned} \chi_{1\mathbf{x}} &= \partial_i A^i(\mathbf{x}), \\ \chi_{2\mathbf{x}} &= \partial_i \Pi^i(\mathbf{x}) + J^0. \end{aligned}$$

Hence we compute the  $C$  matrix elements

$$C_{1\mathbf{x}, 2\mathbf{y}} = -C_{2\mathbf{y}, 1\mathbf{x}} = -\nabla^2 \delta^3(\mathbf{x} - \mathbf{y}),$$

while all the others are vanishing. The inverse matrix elements (basically the Green functions) are given by

$$(C^{-1})_{1\mathbf{x}, 2\mathbf{y}} = -(C^{-1})_{2\mathbf{y}, 1\mathbf{x}} = -\int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

The non-vanishing Poisson brackets are

$$\begin{aligned} [A^i(\mathbf{x}), \chi_{2\mathbf{y}}]_P &= -\frac{\partial}{\partial x^i} \delta^3(\mathbf{x} - \mathbf{y}), \\ [\Pi_i(\mathbf{x}), \chi_{1\mathbf{y}}]_P &= \frac{\partial}{\partial x^i} \delta^3(\mathbf{x} - \mathbf{y}). \end{aligned}$$

The non-vanishing Dirac bracket is given by

$$[A^i(\mathbf{x}), \Pi_j(\mathbf{y})]_D = \delta_j^i \delta^3(\mathbf{x} - \mathbf{y}) + \frac{\partial^2}{\partial x^i \partial x^j} \left( \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \right).$$

After quantization, this is essentially the position representation of the previous commutator result.

## 4 Conclusions

In complicated systems, especially when there exists gauge symmetries, constraints are useful in obtaining a Hamiltonian description of these systems. The Dirac bracket formalism is a useful framework to consistently quantize these constrained systems, as least for the theories that we are very familiar with.

## References

- [1] P. A. M. Dirac, *Generalized Hamiltonian Dynamics*, Canadian Journal of Mathematics 2 (1950).
- [2] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series 2, chapter 1 and 2, Belfer Graduate School of Science, New York (1964).
- [3] S. Weinberg, *Quantum Theory of Fields, Vol. 1 Foundations*, section 7.6 and 8.3, Cambridge University Press (1995).
- [4] M. Henneaux, C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (1992).
- [5] H. J. Rothe, K. D. Rothe, *Classical and Quantum Dynamics of Constrained Hamiltonian Systems*, World Scientific Publishing Company (2010).
- [6] P. K. Townsend, *Part III Lectures on String Theory*, unofficial notes: [http://www.bartholomewandrews.com/teaching/notes/year5/string\\_theory/Lectures.pdf](http://www.bartholomewandrews.com/teaching/notes/year5/string_theory/Lectures.pdf)